SBU AMS Quantitative Finance Qualifying Exam

August 2020

Your solutions must be submitted within 5 minutes of the end of the exam. Submission instructions:

1. Scan your pages, ordered and oriented appropriately, into a single PDF file. Make sure that each problems solution is clearly labeled.
2. Email the PDF file to

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   Mrs. Christine Rota (Christine.Rota@stonybrook.edu)

   Your email should be titled **AMS QF Exam** and does not need to contain any text in its body.
3. **Late submissions** those received after 13:05 am EDT on August, 19, 2020 (as timestamped by the SBU email service) will not be scored.

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

Name (Print clearly):

Student ID:

Signature:

Detailed Instructions for Taking this Exam Over Zoom

1. This exam is conducted via Zoom on August 19, 2020, from 9:00 am to 13:00 am EDT.
2. The entire Zoom meeting and chat messages are being recorded.
3. This is a closed book, closed note exam.
4. Hand calculators (or other computing devices) may not be used during the exam.
5. You should join the Zoom meeting from two devices: Your computer/laptop/tablet (with webcam), and your smartphone (with camera).
6. Audio should be muted, and video must be kept on during the exam.
7. Your computer webcam must fully show your face; your smartphone camera should show your hands and workspace, including the pages of paper being used for the exam.
8. At the very beginning of the exam, during set up, you will be asked to do a brief environment scan, showing the workspace where your computer is and the desk/table/floor where you will be writing your work.
9. You are required to bring enough blank pieces of paper to use for the exam. You will show the blank pages at the beginning, during the environment scan on Zoom.
10. You are not allowed to use the internet for any searching or communication with others, with the sole exception of communication with the proctor(s) via Zoom chat (which is set so that your chats only go privately to hosts, not to others).
11. Use a clean piece of paper to answer each question. Multiple pages for one problem may be used, if necessary. Clearly label all pages.
12. No students are allowed to leave the Zoom meeting until the exam is over.
13. If you finish the exam early, then submit your exam and remain in the Zoom meeting until the conclusion of the exam at 13:00 am EDT.
14. After submitting your exam, you can study for another exam or work on anything else, while staying in view in the Zoom meeting.
15. There are four parts of the exam:
    - Part 1: Do 2 out of problems 1, 2, 3. (AMS511)
    - Part 2: Do 2 out of problems 4, 5, 6. (AMS512)
    - Part 3: Do 2 out of problems 7, 8, 9. (AMS513)
    - Part 4: Do 2 out of problems 10, 11, 12. (AMS517)

Eight problems to be graded: ______________________________

1. **Interest rate and bonds**

   (1) The 6-month and 1-year rates of zero-coupon bonds are both 10% per annum. For a bond that has a life of 18 months and pays a coupon of 8% per annum (with semiannual payments and one having just been made), the yield is 10.4% per annu m. What is the bond's price? What is the 18-month zero rate? All rates are quoted with semiannual compounding.

   (2) A 10-year 8% coupon bond currently sells for \$90. A 10-year 4% coupon bond currently sells for \$80. What is the 10-year zero rate?

2. **Futures and put-call parity**

   (1) Suppose that F_1 and F_2 are two futures contracts on the same commodity with times to maturity, t_1 and t_2, where $t_2 > t_1$. Prove that

   $$F_2 \leq F_1 e^{r(t_2 - t_1)},$$

   where r is the interest rate (assumed) constant) and there are no storage costs. For the purpose of this problem, assume that a futures contract is the same as a forward contract.

   (2) Denote by K the strick price, T the time to maturity, r the risk-free interest rate, D the dividents that are expected to be paid, S_0 the current stock price. Let c and p be the European call and put option prices, respectively. Show the put-call parity $c + D + Ke^{-rT} = p + S_0$ by constructing equavalent portfolios.

3. **Binomial tree methods**

   The volatility of a non-dividend-paying stock whose price is \$78, is 30%. The risk-free rate is 3% per annum (continuously compounded) for all maturities.

   (1) Calculate the risk-neutral probability p when a 2-month time step is used.

   (2) What is the value a 4-month European call option with a strike price of \$80 given by a two-step binomial tree.

   (3) Suppose a trader sells 1,000 options (10 contracts). What position in the stock is necessary to hedge the traders position at the time of the trade?

4. **Mean-variance portfolios**

   You have one unit of capital available and are a mean-variance optimizer. Consider p assets and let $\boldsymbol{\mu}$ denote their return mean vector, $\boldsymbol{\Sigma}$ the return covariance matrix and $\mathbf{w}$ the weight vector of p assets. Assume that assets can be shorted.

   (1) Formulate a quadractic program whose solutions represent the mean-variance efficient set of portfolios and solve the optimal weights for the efficient portfolio.

   (2) Given any two efficient mean-variance portfolios with weights $\mathbf{w}_1$ and $\mathbf{w}_2$, respectively, shown that for any $\alpha \in (0, 1)$, the new portfolio with weight $\alpha\mathbf{w}_1 + (1 - \alpha)\mathbf{w}_2$ is an efficient mean-variance portfolio.

5. **The CAPM**

   You have one unit of capital available and are a mean-variance optimizer. Consider p assets and let $\boldsymbol{\mu}$ denote their return mean vector, $\boldsymbol{\Sigma}$ the return covariance matrix,

r_f the risk-free interest rate, and $\mathbf{w}$ the weight vector of p assets. Assume that assets can be shorted.

(1) Formulate a quadractic program whose solutions represent the mean-variance efficient set of portfolios.

(2) Show that there exists a fixed market portfolio such that for any target returns, the weights of p assets in the efficient portfolio are proporational to those of p assets in the market portfolio.

6. **Marchenko-Pastur Distribution**

You are given the returns of $N = 150$ assets over $T = 250$ time periods.

(1) Compute the parameter q for the Marchenko-Pastur distribution of eigenvalues for a correlation matrix of uncorrelated assets for an estimation problem of this type.

(2) Compute the lower and upper bound for the associated Marchenko-Pastur distribution given q.

(3) You are given the partial list of sorted eigenvalues of the sample correlation matrix: $\{16.2, 8.2, 6.2, 3.1, 4.8, 2.2, 6.8, 1.6, 2.5, 1.4, 2.3, ...\}$. Based solely on the distribution (without any adjustment for sample size), which eigenvalues appear to be statistically meaningful?

7. **Brownian motion**

Suppose $B_1(t)$ and $B_2(t)$ are Brownian motions and

$$dB_1(t)dB_2(t) = \rho(t)dt,$$

where $\rho(t)$ is a stochastic process taking values strictly between -1 and 1. Define processes $W_1(t)$ and $W_2(t)$ such that $B_1(t) = W_t(t)$ and

$$B_2(t) = \int_0^t \rho(s)dW_1(s) + \int_0^t \sqrt{1-\rho^2(s)}dW_2(s),$$

and show that $W_1(t)$ and W_2 are independent Brownian motions.

8. **Vasicek model for interest rates**

Consider the Vasicek model $dx_t = \theta(\mu - x_t)dt + \sigma dW_t$, where W_t is a standard Brownian motion with $W_0 = 0$. Show that for $t > 0, s > 0$,

$$\mathrm{Cov}(x_t, x_s) = \frac{\sigma^2}{2\theta}\left(e^{-\theta|t-s|} - e^{-\theta(t+s)}\right).$$

9. **Black-Scholes pricing theory**

Suppose the stock prices S_1 and S_2 follow geometric Brownian motions, $dS_{i,t} = \mu_i S_{i,t}dt + \sigma_i S_{i,t}dW_{i,t}$, $i = 1, 2$, where μ_i and σ_i are the drift and volatility of the process, and $W_{i,t}$ are two independent standard Brownian motions with $W_{i,0} = 0$. Assuming constant interest rate r, perfectly divisible securities, zero dividends and no transaction costs. Consider a European type financial claim that pays $\max\{\frac{S_{1,T}}{S_{2,T}} - K, 0\}$ at maturity T. (1) Derive the differential equation for the price of the option. (2) What is the price of the financial claim at time t?

10. **Tail behavior of Gaussian copulas**

    (X_1, X_2) has a bivariate Gaussian copula with correlation ρ. Show that the low tail-dependence coefficient of (X_1, X_2) is zero, hence the Gaussian copula is asymptotically independent in tails.

11. **Gumbel's bivariate logistic distribution** Let X and Y be random variables with a joint distribution function given by

$$H(x, y) = (1 + e^{-x} + e^{-y})^{-1}, \qquad x, y \in \mathbb{R}.$$

    Show that (1) X and Y have standard univariate logistic distribution,

$$F(x) = \frac{1}{1 + e^{-x}}, \qquad G(y) = \frac{1}{1 + e^{-y}}$$

    (2) the copula of X and Y is given by

$$C(u, v) = \frac{uv}{u + v - uv}$$

12. **VaR and ES of ARMA-GARCH models**

    Suppose returns y_t of an asset follow the AR(1)-GARCH(1,1) model ($a \in (-1, 1), a \neq 0, \alpha > 0, \beta > 0, \alpha + \beta < 1$)

$$y_t = ay_{t-1} + z_t, \qquad z_t = \sigma_t \epsilon_t, \qquad \sigma_t^2 = \omega + \alpha z_{t-1}^2 + \beta \sigma_{t-1}^2,$$

    where ϵ_t are independent and identically distributed standard normal random variables with mean 0 and variance σ^2. Compute 95% 1-day and 5-day VaR and ES for the long position of the asset?